Bounds on the Location of the Maximum Stirling Numbers of the Second Kind

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Abstract

Let $S(n,k)$ denote the Stirling number of the second kind, and let K_n be such that

$$S(n, K_n - 1) < S(n, K_n) \geq S(n, K_n + 1).$$

Using a probabilistic argument, we show that, for all $n \geq 2$,

$$\lfloor e^{w(n)} \rfloor - 2 \leq K_n \leq \lfloor e^{w(n)} \rfloor + 1,$$

where $\lfloor x \rfloor$ denotes the integer part of x, and $w(n)$ denotes Lambert's W function.

1 Introduction

The Stirling number of the second kind, denoted $S(n,k)$, plays a fundamental role in many combinatorial problems. It counts the number of partitions of $\{1, \ldots, n\}$ into k non-empty, pairwise disjoint subsets, and may be defined recursively as

$$S(n,k) = S(n-1, k-1) + kS(n-1, k), \quad n \geq 1, \ k \geq 1,$$

together with $S(0,0) = 1, \ S(n,0) = 0, \ n \geq 1$.

According to Harper [8], for each $n \geq 1$, the polynomial $\sum_{k=0}^{n} S(n,k) x^k$ has only real zeros. By Newton's inequalities ([7], p. 52), $\log S(n,k)$ is strictly concave in k. It follows that there exists some $1 \leq K_n \leq n$ such that

$$S(n,1) < \ldots < S(n, K_n) \geq S(n, K_n + 1) > \ldots > S(n,n).$$

In other words, the sequence $S(n,k),\ k=1,\dots,n$, is unimodal, K_n being a unique mode if $S(n,K_n) \neq S(n,K_n+1)$.

Determining the value of K_n is an old problem ([9, 10, 6, 1, 5, 11, 15, 13, 2]). A related long-standing conjecture ([15, 3, 12]) is that there exists no $n>2$ such that $S(n,K_n)=S(n,K_n+1)$. See [3] for a historical sketch and recent developments.

In particular, Canfield and Pomerance [3] noted that

$$K_n \in \{\lfloor e^{w(n)} \rfloor - 1,\ \lfloor e^{w(n)} \rfloor\} \tag{1}$$

for both $2 \le n \le 1200$ and n large enough (no specific bound is known on how large n has to be; see also [2]). Here and in what follows, $\lfloor x \rfloor$ denotes the integer part of x and $w(n)$ is Lambert's W function defined by

$$n = w(n)e^{w(n)}.$$

Based on this, it seems likely that (1) holds for all n. The purpose of this note is to present the following non-asymptotic bounds.

Theorem 1.

$$\lfloor e^{w(n)} \rfloor - 2 \le K_n \le \lfloor e^{w(n)} \rfloor + 1, \quad n \ge 2. \tag{2}$$

Theorem 1 can be compared with the non-asymptotic bounds of Wegner [15]:

$$K_n < \frac{n}{\log n - \log\log n}, \quad n \ge 3; \tag{3}$$

$$K_n > \frac{n}{\log n}\left(1 + \frac{\log\log n - 1}{\log n}\right), \quad n \ge 31. \tag{4}$$

Note that the upper and lower bounds in (2) differ by 3, whereas the difference between the upper bound (3) and the lower bound (4) tends to ∞ as $n \to \infty$. More precisely, it can be shown (details omitted) that the upper bound in (2) implies (3) if $n \ge 7$, and the lower bound in (2) implies (4) if $n \ge 34$.

In Section 2 we prove (2) using a probablistic result of Darroch [4]. The possibility of further refinements is discussed in Section 3.

2 Proof of (2)

Recall Dobinski's formula

$$e^x \sum_{k=1}^{n} S(n,k)x^k = \sum_{k=1}^{\infty} \frac{k^n x^k}{k!}, \quad n \geq 1. \tag{5}$$

In particular

$$e \sum_{k=1}^{n} S(n,k) = \sum_{k=1}^{\infty} \frac{k^n}{k!}. \tag{6}$$

Dividing (5) by (6) we get

$$\left(\sum_{k=0}^{\infty} \frac{1}{ek!} x^k\right) \sum_{k=1}^{n} \frac{S(n,k)}{\sum_{i=1}^{n} S(n,i)} x^k = \sum_{k=1}^{\infty} \frac{k^n/k!}{\sum_{i=1}^{\infty} i^n/i!} x^k.$$

This has the following interpretation. If we let S be a random variable with probability mass function (pmf) $\Pr(S = k) = S(n,k)/\sum_{i=1}^{n} S(n,i)$, $k = 1, \ldots, n$, and let Z be a Poisson(1) random variable independent of S, then the pmf of $S + Z$ is

$$\Pr(S + Z = k) = \frac{k^n/k!}{\sum_{i=1}^{\infty} i^n/i!}, \quad k = 1, 2, \ldots$$

While the mode of S is hard to determine, that of $S + Z$ is straightforward. (As usual, we call a random variable X on $\{0, 1, \ldots\}$ unimodal if its pmf is unimodal, and call any mode of the pmf a mode of X.) To relate the mode of S to that of $S + Z$, we invoke a classical result of Darroch [4] (see Pitman's survey [14]). Note that S can be written as a sum of n independent Bernoulli random variables since the polynomial $\sum_{k=1}^{n} S(n,k)x^k$ has only real zeros.

Theorem 2 ([4])**.** *Let X_i, $i = 1, \ldots, n$, be independent Bernoulli random variables, i.e., each X_i takes values on $\{0, 1\}$. Then for any mode m of $S = \sum_{i=1}^{n} X_i$*

$$|m - ES| < 1.$$

As a consequence of Theorem 2, we have

Proposition 1. *Let $S = \sum_{i=1}^{n} X_i$ be a sum of independent Bernoulli random variables. Let Z be a* Poisson(1) *random variable independent of S. Assume $S + Z$ has a unique mode m_1, and denote any mode of S by m_0. Then*

$$m_0 \leq m_1 \leq m_0 + 2. \tag{7}$$

Proof. Note that, since the pmfs of S and Z are both log-concave, the pmf of $S+Z$ is log-concave and hence unimodal. Denote $\mu = ES$. By Darroch's rule, $|\mu - m_0| < 1$. We show that Darroch's rule applies to $S+Z$, i.e., $|\mu + 1 - m_1| < 1$. The claim then readily follows. Let $Z_k,\ k \geq 2$, be Binomial$(k, 1/k)$ random variables, independent of S. Then $S+Z_k$ is a sum of independent Bernoullis for which Darroch's rule applies; if we let m_k be a mode of $S+Z_k$, then $|\mu+1-m_k| < 1$. Moreover, assuming m_1 is the unique mode of $S+Z$, we have $\lim_{k\to\infty} m_k = m_1$. Thus $|\mu + 1 - m_1| < 1$. $\square$

On the other hand, we have

Proposition 2. *For $n \geq 2$, the sequence $k^n/k!,\ k = 1, 2, \ldots,$ is unimodal with a unique mode at either $k = \lfloor e^{w(n)} \rfloor$ or $k = \lfloor e^{w(n)} \rfloor + 1$.*

Proof. Denote $u = e^{w(n)}$ and consider the ratio

$$f(k) = \frac{(k+1)^n/(k+1)!}{k^n/k!} = \frac{(k+1)^{n-1}}{k^n}.$$

It is easy to see that $f(k) \neq 1$ for all $k \geq 1$. We also show that $f(k) > 1$ if $k < u - 1$ (i.e., $k \leq \lfloor u \rfloor - 1$) and $f(k) < 1$ for $k > u$ (i.e., $k \geq \lfloor u \rfloor + 1$). The claim then follows.

Noting that $f(k)$ decreases in k, we only need to show $f(u-1) > 1$ and $f(u) < 1$. However, direct calculation gives

$$\begin{aligned} \log f(u-1) &= -w(n) - n\log\left(1 - e^{-w(n)}\right) \\ &> -w(n) - n\left(-e^{-w(n)}\right) = 0; \\ \log f(u) &= n\log\left(1 + e^{-w(n)}\right) - \log\left(e^{w(n)} + 1\right) \\ &< ne^{-w(n)} - \log\left(e^{w(n)}\right) = 0. \quad \square \end{aligned}$$

Then we obtain (2) as a consequence of Propositions 1 and 2.

Corollary 1. *Let $n \geq 2$, and denote $k_* = \lfloor e^{w(n)} \rfloor$. If $k_*^n/k_*! > (k_*+1)^n/(k_*+1)!$, then $k_* - 2 \leq K_n \leq k_*$; otherwise $k_* - 1 \leq K_n \leq k_* + 1$. At any rate (2) holds.*

3 Discussion

A natural question is whether Corollary 1 can be further improved using this argument. This leads to an investigation of the bounds in (7). It turns out that the lower bound in (7) is

achievable. For example, in the setting of Proposition 1, if we let $n = 2$ and $\Pr(X_i = 1) = 1 - \Pr(X_i = 0) = p_i,\ i = 1, 2,$ with $p_1 = 1/3$ and $p_2 = 2/5$, then $m_0 = m_1 = 1$ by direct calculation. It seems difficult, however, to find an example where the upper bound in (7) is achieved. After some experimentation we suspect that this upper bound is not achievable. This is further supported by the fact that, in the setting of Proposition 1, we always have $m_1 \leq m_0+1$ when $n \leq 5$. To show this, let $c_i = \Pr(S = i),\ i = 0, 1, \ldots$. By Newton's inequalities

$$c_{i+1}^2 \geq \frac{(i+2)(n-i)}{(i+1)(n-i-1)} c_i c_{i+2}, \quad 0 \leq i \leq n-2.$$

When $n \leq 5$ and $0 \leq i \leq n-2$ we have

$$\frac{(i+2)(n-i)}{(i+1)(n-i-1)} \geq 2.$$

Thus $c_{i+1}^2 \geq 2c_i c_{i+2}$ and $c_{m_0+1}^2 \geq 2c_{m_0} c_{m_0+2}$ in particular. Since m_0 is a mode of S, we know $c_{m_0} \geq c_{m_0+1}$. Thus

$$c_{m_0} \geq 2c_{m_0+2}.$$

However, a simple calculation gives

$$\begin{aligned} e[\Pr(S+Z = m_0+1) - \Pr(S+Z = m_0+2)] &= \sum_{k=0}^{m_0} \frac{c_k}{(m_0-k)!(m_0+2-k)} - c_{m_0+2} \\ &\geq \frac{c_{m_0}}{2} - c_{m_0+2} \geq 0, \end{aligned}$$

which rules out $m_1 = m_0 + 2$ under the assumption that m_1 is the unique mode of $S + Z$.

Conjecture 1. *In the setting of Proposition 1,* $m_0 \leq m_1 \leq m_0 + 1$.

It is clear that Conjecture 1 implies a sharper version of (2)

$$\lfloor e^{w(n)} \rfloor - 1 \leq K_n \leq \lfloor e^{w(n)} \rfloor + 1;$$

this is tantalizingly close to proving (1) for all $n \geq 2$.

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